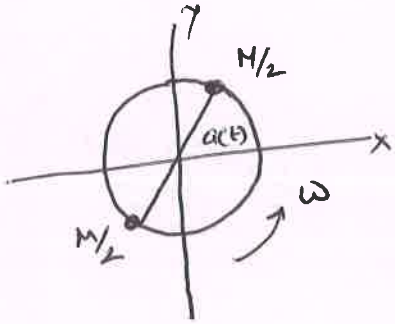
Lecture 5   IUCAA Refresher Course   May-19-25   SK

Basics of GW Theory

Coalescing Binary - Newtonian model with GR input.



The basic idea here is to allow the 'a' (distance between the two objects) to become a function of time. If a decreases as time progresses, at some 't' $(t = t_c)$ the objects will coalesce. How does the emitted GW signal show this coalescence. We work out a simple Newtonian model, essentially based on the eqn $\frac{dE}{dt} = -L_{GW}$ assuming the $a(t)$ to be time-dependent. For L_{GW}, we take the expression $L_{GW} = \frac{2}{5}\frac{GM^2a^4\omega^6}{c^5}$.

As before, we have individual masses as $M/2$ and total mass equal to M.

What is to K.E T, the P.E. V and the total energy E.

$$T = 2\left(\frac{1}{2}\frac{M}{2}\omega^2R^2\right) = \left(\frac{1}{4}M\omega^2\frac{a^2}{4}\right)2 = \frac{1}{8}M\omega^2a^2$$

$$V = -\frac{G\,\frac{M}{2}\,\frac{M}{2}}{a} = -\frac{GM^2}{4a}.$$

Kepler's 3rd law says $\omega^2a^3 = GM$.

Thus $T = \frac{GM^2}{8a}$

Hence $E = \frac{GM^2}{8a} - \frac{GM^2}{4a} = -\frac{GM^2}{8a}.$

Since $\frac{dE}{dt} \neq 0$ and we take it as $-L_{GW}$, we have

(2)

$$\frac{dE}{dt} = -\frac{2}{5}\frac{GM^2a^4\omega^6}{c^5}$$

$$\frac{dE}{dt} = \frac{d}{dt}\left[-\frac{GM^2}{8a}\right] = \frac{GM^2}{8a^2}\dot{a}$$

Hence $$\frac{GM^2}{8a^2}\dot{a} = -\frac{2}{5}\frac{GM^2a^4\omega^6}{c^5}$$

$$= -\frac{2}{5}\frac{GM^2.(a^9\omega^6)}{c^5a^5} = -\frac{2}{5}\frac{(GM^2).(GM)^3}{c^5a^5}$$

$$\therefore \quad a^3\dot{a} = -\frac{16}{5}\frac{(GM)^3}{c^5}$$

$$\frac{1}{4}\frac{d}{dt}a^4 = -\frac{16}{5}\frac{(GM)^3}{c^5}$$

$$\Rightarrow \quad \frac{1}{4}a^4 = -\frac{16}{5}\frac{(GM)^3}{c^5}.t + C_1$$

At $t = t_c$ $\quad a = 0$

$$C_1 = \frac{16}{5}\frac{(GM)^3}{c^5}t_c.$$

$$\therefore \quad a^4(t) = \frac{64}{5}\frac{(GM)^3}{c^5}(t_c - t)$$

At $t=0$ $\quad a = a_0$ $\quad a_0^4 = \frac{64(GM)^3}{5c^5}t_c.$

$$\therefore \quad t_c = \frac{5c^5a_0^4}{64(GM)^3}$$ time of Coalescence/merger.

Freq at $t=0$ is $a_0^3\omega_0^2 = GM$ $\quad \omega_0^2 = \left(\frac{GM}{a_0^3}\right).$

$$\omega_0 = \sqrt{\frac{GM}{a_0^3}} \Rightarrow \nu_0 = \frac{1}{2\pi}\sqrt{\frac{GM}{a_0^3}}$$

Therefore,

$$a(t) = \left[\frac{64}{5}\frac{(GM)^3}{c^5}\right]^{1/4} (t_c - t)^{1/4}$$ ③

and $$\omega(t) = \sqrt{\frac{GM}{a^3(t)}}.$$

What is the GW signal?

The amplitude part now contains $a(t)$ and is time-dependent.

We need to find the phase part.

$$\phi(t) = -\int_t^{t_c} \omega(t)\,dt = -\int_t^{t_c} \frac{\sqrt{GM}}{a^{3/2}}\,dt.$$

$$= -\int_t^{t_c} \frac{\sqrt{GM}\;dt}{\left[\frac{64}{5}\frac{(GM)^3}{c^5}(t_c - t)\right]^{3/8}}$$

$$\phi(t) - \phi_c = -\int_t^{t_c} \omega(t)\,dt$$

Thus $$\phi(t) - \phi(t_c) = \phi(t) - \phi_c.$$

$$= \sqrt{GM}\left(\frac{5c^5}{64(GM)^3}\right)^{3/8} \frac{(t_c - t)^{-3/8+1}}{-3/8+1}\Bigg|_t^{t_c}.$$

$$= \sqrt{GM}\left(\frac{5c^5}{64(GM)^3}\right)^{3/8} \frac{8}{5}\,(t_c - t)^{5/8}.\Bigg|_t^{t_c}$$

$$= -(GM)^{1/2 - 9/8}\, 5^{-5/8}\, c^{15/8}\, 2^{3 - 18/8}\, (t_c - t)^{5/8}$$

$$= -\left(\frac{c^3}{5GM\,2^{-6/5}}\right)^{5/8} (t_c - t)^{5/8}$$

$$\therefore\ \phi(t) = \phi_c - \left(\frac{c^3}{5G\mathcal{M}}\right)^{5/8} (t_c - t)^{5/8}.$$

$$\mathcal{M} = 2^{-6/5} M \longrightarrow \text{chirp mass.}$$

Hence, the signal $h_+(t)$ is

$$h_+(t) = -\left(\frac{GM}{c^2 r}\right)\left(\frac{GM}{c^2 a(t)}\right) \cos 2\phi(t).$$ (4)

where $$a(t) = \left(\frac{64 (GM)^3}{5c^5}\right)^{1/4} (-t + t_c)^{1/4}.$$

$$\phi(t) = \phi_c - \left(\frac{c^3}{5G\mathcal{M}}\right)^{5/8} (-t + t_c)^{5/8}.$$

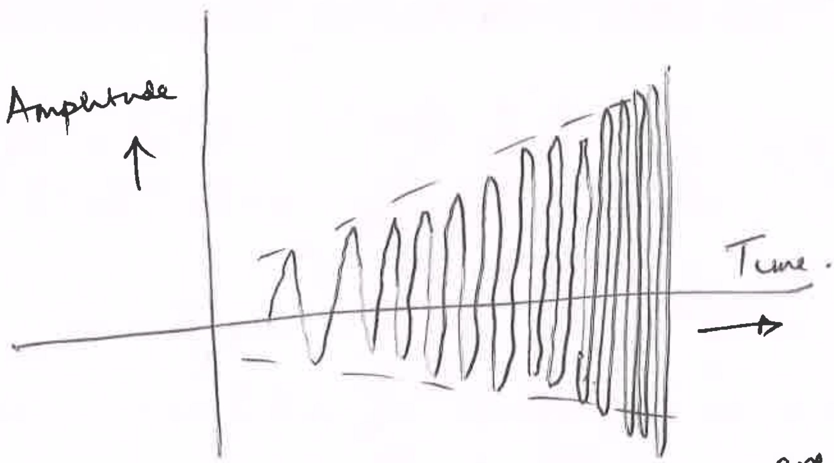
and $\mathcal{M} = 2^{-6/5} M$ is the chirp mass; $t_c = \frac{5c^5 a_0^4}{64 (GM)^3}$; $\omega_0^2 a_0^3 = GM$

Take two 1 solar mass ($M_\odot$) objects $M = 2M_\odot$

Assume $\nu = 10$ Hz (orbital). We can calculate t_c and a_0

$a_0 = 4.073 \times 10^{15}$ m $\qquad t_c = 275.18$ s,

The nature of the waveform is



This is known as the chirp. [freq increases as $a \downarrow$ $t \uparrow$]

Note at $t = t_c$ amplitude becomes infinite.

Thus, within this simple model, we are able to generate the pre-merger signal seen in GW150914.

The merger cannot be described in this model.

The post-merger is understood using BH perturbation theory

One can improve the pre-merger signal going beyond the Newtonian approximation.

Hulse-Taylor Binary Pulsar and GW

Binary pulsar $\rightarrow$ GW radiation $\rightarrow$ periastron shift.

Calculate the cumulative periastron shift.

Reported by Hulse, Taylor in 1973 for PSRB 1913+16

Reported for many other binary pulsars later on, including recently.

The analysis can be easily done using whatever we have learnt till now.

$$\omega^2 a^3 = GM.$$

Time period $P = \frac{2\pi}{\omega}$

$$\therefore \quad a^3 \frac{4\pi^2}{P^2} = GM.$$

$$\therefore \quad 3\frac{\dot{a}}{a} = 2\frac{\dot{P}}{P} \longrightarrow (1)$$

Energy $E = -\frac{GM^2}{8a}$ $\qquad \dot{E} = +\frac{GM^2}{8a^2}\dot{a}$

$$\therefore \quad \frac{\dot{E}}{E} = -\frac{\dot{a}}{a} = -\frac{2}{3}\frac{\dot{P}}{P}.$$

Now $\dot{E} = \frac{dE}{dt} = -\frac{2}{5}\frac{GM^2a^4\omega^6}{c^5}$

$$\therefore \quad \frac{-\frac{2}{5}\frac{GM^2a^4\omega^6}{c^5}}{-GM^2/8a} = -\frac{2}{3}\frac{\dot{P}}{P}$$

$$\frac{16}{5}\frac{a^5\omega^6}{c^5} = -\frac{2}{3}\frac{\dot{P}}{P}$$

$$\therefore \quad \frac{\dot{P}}{P} = -\frac{48}{10}\frac{a^5\omega^6}{c^5} = -\frac{24}{5}\frac{(GM)^3}{c^5 a^4}.$$

$$\frac{\dot{P}}{P} = -\frac{24}{5}\frac{(GM)^3}{a^4c^5}$$

$$\Rightarrow \quad \dot{P} = -\frac{24}{5}\cdot\frac{(GM)^3}{a^4c^5}\cdot\frac{2\pi}{\omega}$$

$$= -\frac{48}{5}\frac{(GM)^3}{a^4c^5\,\omega}$$

$$\omega^2a^3 = GM$$
$$a^4 = \left(\frac{GM}{\omega^2}\right)^{4/3}$$

$$= -\frac{48}{5}\frac{(GM)^3}{c^5\cdot\left(\frac{GM}{\omega^2}\right)^{4/3}}\cdot\frac{1}{\omega}$$

$$= -\frac{48}{5c^5}(GM)^{5/3}\cdot\omega^{5/3} = -\frac{48}{5}\left(\frac{GM\omega}{c^3}\right)^{5/3}$$

Values for PSR B 1913+16.

Period 7.75 hrs. $= 7.75\times 3600 = 27907$ s. (27907)s

$M = 2.8\,M_\odot$ $\quad a = 1944\times10^6$ m

Eccentricity $e = 0.667$ $\quad f(e) = \dfrac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{7/2}} = 11.84$

(Peters-Mathews factor).

This needs to be multiplied to get the correct result for an eccentric orbit.

Hence, using all these values

$$\dot{P} = -2.42\times10^{-12}$$

In 1 year, the amount is $75\,\mu s$.

What is the cumulative periastron shift?

Cumulative periastron shift

Initial period say P_0 $\qquad \nu = \dot{P}$

Change after in period $P_0 - \nu P_0$

after 2 periods $P_0 - 2\nu P_0$.

After N periods cumulative value is

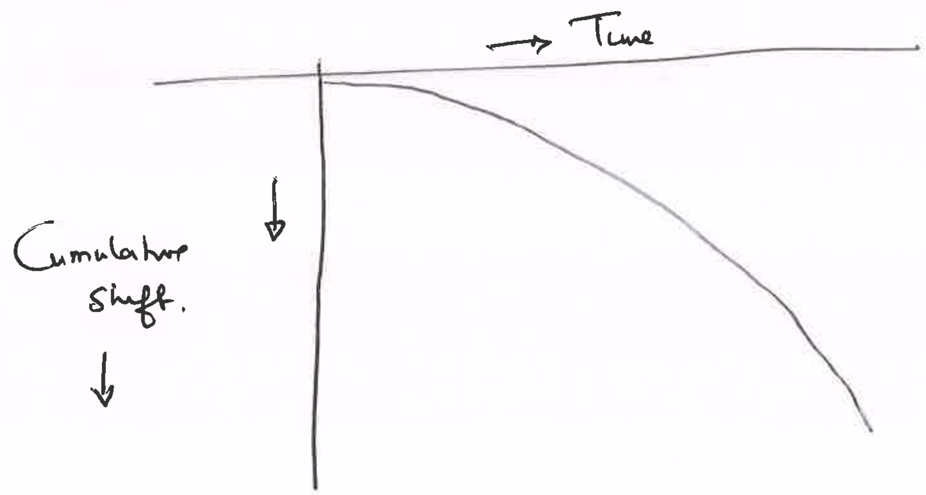
$$N P_0 - \nu \cdot \frac{N(N+1)}{2} P_0 .$$

$$\text{Shift} = \cancel{NP_0} - \nu \frac{N(N+1)}{2} P_0 - \cancel{NP_0}$$

$$= -\nu \frac{N(N+1)}{2} P_0 .$$

Graph : Time elapsed in N revolutions $N = t/P_0$.

Hence cumulative shift $= -\nu \dfrac{t/P_0\,(t/P_0 + 1)}{2}$.



Nature of graph.

Tallies very well with observations. (High accuracy).

Seen also for PSR - J0737 - 3039

(see Tutorial 5).

Obvious questions

1. Can GW signals from binary pulsars be seen at LIGO ?

2. Do such binary pulsars coalesce – have we seen or may see any such event ?

3. What is the coalescence time for PSR B 1913+16 ? Can we ever see it coalesce ?

(See Tutorials for answers)

Summary

1. $g_{ij} = \eta_{ij} + h_{ij}$, $\bar{h}_{ij} = h_{ij} - \frac{1}{2}\eta_{ij} h$.

Gauge condition $\partial^i \bar{h}_{ij} = 0$. de Donder/Lorentz.

Linearised Einstein eqn $\Box \bar{h}_{ij} = -\frac{16\pi G}{c^4} T_{ij}$

2. TT gauge $h_{00} = h_{0\alpha} = 0$, $h = 0$, $\partial^\alpha h_{\alpha\beta} = 0$.

2 radiative degrees of freedom $h_{\alpha\beta}^{TT}$

3. General prescription to obtain $h_{\alpha\beta}^{TT}$ when $\bar{h}_{ij}$ obeys the Lorentz gauge/de Donder gauge and so does $\bar{h}_{ij}'$.

We have gauge fn $\xi_i = B_i e^{-i k_m x^m}$ and A_{ij}, A_{ij}' as the amplitudes of $\bar{h}_{ij}$ and $\bar{h}_{ij}'$.

TT gauge $A'_{\alpha\beta} = \left(P_\alpha^\gamma P_\beta^\delta - \frac{1}{2} P_{\alpha\beta} P^{\gamma\delta}\right) A_{\gamma\delta}$.

One can find B_i in terms of A_{ij} and n^α

4. (a) Geodesic eqn in TT gauge to lowest order $\rightarrow \frac{d^2 x^\alpha}{dt^2} = 0$

x^α do not see the 'h'.

(b) Deviation sees the 'h' through δL_α $[L_\alpha = L_0 + \delta L_\alpha]$.

$\delta L_\alpha = \frac{1}{2} \ddot{h}_{\alpha\beta} L^\beta$. $\lambda_{GW} \gg L$

(c) Ring of particles changes shape to ellipse and reverts back periodically to circle and so on. Memory.

(d) Proper length of a rod along 'x' in the presence of a GW along 'z' gives $\frac{\Delta L}{L_0} = \frac{1}{2} h_+$. and so on.

(e) Signal $h(t) = F_+ h_+ + F_\times h_\times$ $D_{\alpha\beta} = \frac{1}{2}\left(n_\alpha^{(1)} n_\beta^{(1)} - n_\alpha^{(2)} n_\beta^{(2)}\right)$

where $F_+ = D^{\alpha\beta} e_{\alpha\beta}^+$ $F_\times = D^{\alpha\beta} e_{\alpha\beta}^\times$. $e_{\alpha\beta}^+, e_{\alpha\beta}^\times \rightarrow$ polarisation tensors,

Antenna Pattern fns.

5. Sources $\rightarrow$ $T_{ij} \neq 0$.

Find waveform by Solving $\Box \bar{h}_{ij} = -\frac{16\pi G}{c^4} T_{ij}$

This gives $h^{TT}_{\alpha\beta} = \frac{2G}{c^4 r} \ddot{I}^{TT}_{\alpha\beta}$. 'far zone'.

Can be matched to near-zone which uses multipole expansion with quadrupole defined using $\mathcal{I}_{\alpha\beta}$. (traceless).

6. Defn of energy $T^{GW}_{00} = -\left\langle \partial_0 h^{TT}_{ij} \partial_0 h^{ij\,TT} \right\rangle$.

Avging scale 'L', $\lambda_{GW} < \underline{\underline{L}} < R$

Short wavelength, high freq. approx. $\rightarrow$ 'approx gauge inv' and conserved. stress energy.

7. Luminosity $L_{GW} = \frac{G}{5c^5} \left\langle \dddot{\mathcal{I}}_{\alpha\beta} \dddot{\mathcal{I}}^{\alpha\beta} \right\rangle$ $\leftarrow$ general formula.

Application: equal mass binary $L_{GW} = \frac{2}{5} \frac{G M^2 a^4 \omega^6}{c^5}$.

8. Coalescing binary $\rightarrow$ Newtonian analysis with GR input.

This gives the chirp waveform during the inspiral.

Valid till the merger. Post-merger $\rightarrow$ BH pert. theory.

Merger $\longrightarrow$ Numerical Relativity.

9. Hulse-Taylor binary — cumulative periastron shift.

Formula and plots $\rightarrow$ applicable to similar scenarios.

10. Brief discussions on:

a) History

b) SVT decomposition.

c) Other derivations of quadrupole formula.

d) Ref. for stress-energy tensor (recent).